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Stochastic Lotka–Volterra system with infinite delay

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ABSTRACT

This paper investigates a stochastic Lotka–Volterra system with infinite delay, whose initial data comes from an admissible Banach space C_r. We show that, under a simple hypothesis on the environmental noise, the stochastic Lotka–Volterra system with infinite delay has a unique global positive solution, and this positive solution will be asymptotic bounded. The asymptotic pathwise of the solution is also estimated by the exponential martingale inequality. Finally, two examples with their numerical simulations are provided to illustrate our result.

1. Introduction

Mao et al. [1,2] have investigated the stochastic Lotka–Volterra system

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \dots, x_n(t))\big[(b + Ax(t))\mathrm{d}t + \sigma x(t)\mathrm{d}w(t)\big]. \tag{1.1}$$

They reveal that the environmental noise can suppress a potential population explosion. Namely if the following hypothesis is imposed on the noise

$$\sigma_{ii} > 0 \quad \text{if } 1 \le i \le n \quad \text{whilst} \quad \sigma_{ij} \ge 0 \quad \text{if } i \ne j, \tag{H1}$$

for any $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, the solution of Eq. (1.1) will remain in the positive cone $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0 \quad \text{for all } 1 \le i \le n\}$ with probability 1 and, in particular, it will not explode in a finite time. They also show that the simple hypothesis (H1) on the environmental noise is enough to guarantee the stochastically ultimate boundedness of the solutions of Eq. (1.1). Moreover, the asymptotic pathwise estimation is given. Soon, Bahar et al. [3] find that the stochastic delay Lotka–Volterra system

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \dots, x_n(t))\big[(b + Ax(t-\tau))\mathrm{d}t + \sigma x(t)\mathrm{d}w(t)\big] \tag{1.2}$$

also has the above properties only with the hypothesis (H1) on the noise. However, to the authors' best knowledge, few papers can be found in the literature for these properties of the stochastic Lotka–Volterra system with infinite delay.

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On the other hand, the deterministic Lotka–Volterra model with infinite delay is generally described by the integrodifferential equations

$$\mathrm{d}x(t)/\mathrm{d}t = \mathrm{diag}(x_1(t), \dots, x_n(t))\left[b + Ax(t) + B\int_{-\infty}^{0} x(t+\theta)\mathrm{d}\mu(\theta)\right]. \tag{1.3}$$

There is an extensive literature concerned with the dynamics of this model with infinite delay and we here only mention [4–9]. In [7], Gopalsamy shows that, in order to guarantee the existence and uniqueness and global asymptotic stability of the positive periodic solution of the system (1.3), the coefficient matrices A and B have to satisfy a set of algebraic conditions. However, we are concerned whether there exists the global positive solution for the Lotka–Volterra model (1.3) with infinite delay by taking the environmental noise into account instead of imposing the algebraic conditions on the matrices A and B.

To consider the environmental noise, we stochastically perturb the Lotka–Volterra model (1.3) into the Itô stochastic differential equations with infinite delay

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \dots, x_n(t))\left[\left(b + Ax(t) + B\int_{-\infty}^{0} x(t+\theta)\mathrm{d}\mu(\theta)\right)\mathrm{d}t + \sigma x(t)\mathrm{d}w(t)\right], \tag{1.4}$$

where

$$x = (x_1, \dots, x_n)^{\mathrm{T}}, \qquad b = (b_1, \dots, b_n)^{\mathrm{T}}, \qquad A = (a_{ij})_{n\times n}, \qquad B = (b_{ij})_{n\times n},$$

and $\sigma = (\sigma_{ij})_{n\times n}$ satisfies the conditions (H1), $w(t)$ is a scalar Brownian motion. To avoid the usual well-posedness questions related to functional equations of unbounded delay (see [10,11,9]), we let the initial data $x_0 = \xi$ be positive and belong to the friendly spaces C_r (see [12,10,13]) which defined by

$$C_r := \{\varphi \in C((-\infty, 0]; \mathbb{R}^n_+) : \|\varphi\|_{C_r} = \sup_{-\infty\le s\le 0} \mathrm{e}^{rs}|\varphi(s)| < \infty\},$$

where $r > 0$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \ge 0 \text{ for all } 1 \le i \le n\}$. It is easy to verify that C_r is an admissible Banach space (see [13,9]). And μ is the probability measure on $(-\infty, 0]$ satisfying that

$$\mu_r := \int_{-\infty}^{0} \mathrm{e}^{-2r\theta}\mathrm{d}\mu(\theta) < \infty. \tag{H2}$$

Clearly, the above assumption may be satisfied when $\mu(\theta) = \mathrm{e}^{kr\theta}$ $(k > 2)$ for $\theta \le 0$, so there exists a large number of these probability measures.

This paper is organized as follows: In the next Section, we show that Eq. (1.4) admits a unique solution and the solution will remain in $\mathbb{R}^n_{++}$ with probability one. From the biological point of view, the asymptotic bound properties are more desired than nonexplosion property. We consider them in Section 3. Section 4 discusses the solution of Eq. (1.4) how to vary pathwisely in $\mathbb{R}^n_{++}$. In the last Section, two examples with their numerical simulations are provided to illustrate our result.

2. Global positive solutions

Throughout this paper unless otherwise specified, we use the following notations. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. Let $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i \ge 0 \text{ for all } 1 \le i \le n\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \le i \le n\}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Assume that $w(t)$ is a scalar Brownian motion defined on the complete probability space. If $x(t)$ is an $\mathbb{R}^n$-valued stochastic process on $t \in \mathbb{R}$, we let $x_t = \{x(t+\theta) : \theta \in (-\infty, 0]\}$ for $t \ge 0$.

Xu [14] has proved that, in order for a stochastic functional differential equations with infinite delay to have a unique global solution for any given initial data $\xi \in C_r$, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition. The local Lipschitz condition guarantees that the unique solution exists in $(-\infty, \tau_e]$, where τ_e is the explosion time (see Mao [15]). Clearly, the coefficients of Eq. (1.4) satisfy the local Lipschitz condition, but do not satisfy the linear growth condition.

Theorem 2.1. *Assume that (H1) and (H2) hold. Then, for any system parameters $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and any given positive initial data $x_0 = \xi \in C_r$, there is a unique solution $x(t)$ to Eq. (1.4) on $t \in \mathbb{R}$. Moreover, this solution remains in $\mathbb{R}^n_{++}$ with probability 1, namely $x(t) \in \mathbb{R}^n_{++}$ for all $t \in \mathbb{R}$ almost surely.*

Proof. Since the coefficients of Eq. (1.4) are locally Lipschitz continuous, for any given positive initial data $\xi \in C_r$, there is a unique maximal local solution $x(t)$ on $t \in (-\infty, \tau_e]$, where τ_e is the explosion time. To show that this solution is global, we only need to prove that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large in the sense

$$k_0^{-1} < \min_{-\infty\le\theta\le 0} |\xi(\theta)| \le \max_{-\infty\le\theta\le 0} |\xi(\theta)| < k_0.$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [-\tau, \tau_e) : x_i(t) \notin (k^{-1}, k) \text{ for some } i = 1, 2, \ldots, n\}$$

with usual setting $\inf \emptyset = \infty$,where $\emptyset$ denotes the empty set. Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. If we can prove $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., which implies the desired result. To prove this statement, let us define a C^2-function $V : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ by

$$V(x) = \sum_{i=1}^{n} u(x_i), \tag{2.1}$$

where $u(x_i) = x_i^{0.5} - 0.5\log(x_i)$. Clearly, $u(\cdot) \geq 0$ and $u(0^+) = u(\infty) = \infty$. Let $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, applying the Itô formula to $V(x(t))$ to obtain that

$$\mathrm{d}V\big(x(t)\big) = \mathcal{L}V(x(t), x_t)\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{n} 0.5[x_i^{0.5}(t) - 1]\sigma_{ij}x_j(t)\mathrm{d}w(t),$$

where $\mathcal{L}V : \mathbb{R}^n_{++} \times C_r \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{L}V(x,\varphi) = {} & 0.5\sum_{i=1}^{n}(x_i^{0.5} - 1)\left[b_i + \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}\int_{-\infty}^{0}\varphi_j(\theta)\mathrm{d}\mu(\theta)\right] \\ & + \sum_{i=1}^{n}\big[0.25 - 0.125x_i^{0.5}\big]\left[\sum_{j=1}^{n}\sigma_{ij}x_j\right]^2.\end{aligned} \tag{2.2}$$

We may compute that

$$0.5\sum_{i=1}^{n}\left[x_i^{0.5} - 1\right]\sum_{j=1}^{n} a_{ij}x_j \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}^2\big(x_i^{0.5} - 1\big)^2 + x_j^2\right],$$

and

$$0.5\sum_{i=1}^{n}\big[x_i^{0.5} - 1\big]\sum_{j=1}^{n} b_{ij}\int_{-\infty}^{0}\varphi_j(\theta)\mathrm{d}\mu(\theta) \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left[b_{ij}^2(x_i^{0.5} - 1)^2 + \int_{-\infty}^{0}\varphi_j^2(\theta)\mathrm{d}\mu(\theta)\right]$$

and

$$\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sigma_{ij}x_j\right]^2 \leq \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\sigma_{ij}^2\sum_{j=1}^{n}x_j^2\right] = |\sigma|^2|x|^2.$$

Moreover, by hypothesis (H1),

$$\sum_{i=1}^{n} x_i^{0.5}\left[\sum_{j=1}^{n}\sigma_{ij}x_j\right]^2 \geq \sum_{i=1}^{n}\sigma_{ii}^2 x_i^{2.5}.$$

Substituting the above inequalities into (2.2) yields

$$\mathcal{L}V(x,\varphi) \leq F(x) + n\left[\int_{-\infty}^{0}|\varphi(\theta)|^2\mathrm{d}\mu(\theta) - |x|^2\right],$$

where

$$F(x) = 0.5\sum_{i=1}^{n} b_i\left[x_i^{0.5} - 1\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\big[a_{ij}^2 + b_{ij}^2\big]\big[x_i^{0.5} - 1\big]^2 + (2n + 0.25)|\sigma|^2|x|^2 - 0.125\sum_{i=1}^{n}\sigma_{ii}^2 x_i^{2.5}.$$

It is straightforward to see that $F(x)$ is bounded, say by K, in $\mathbb{R}^n_{++}$. We therefore have

$$\begin{aligned}\mathbb{E}V(x(t)) = {} & \mathbb{E}V(x(0)) + \mathbb{E}\int_0^t \mathcal{L}V(x(s), x_s)\mathrm{d}s \\ \leq {} & \mathbb{E}V(\xi(0)) + Kt + n\sum_{j=1}^{n}\mathbb{E}\int_0^t\left[\int_{-\infty}^{0} x_j^2(s+\theta)\mathrm{d}\mu(\theta) - x_j^2(s)\right]\mathrm{d}s.\end{aligned}$$

By hypothesis (H2), we may compute that

$$\begin{aligned}\int_0^t\int_{-\infty}^0 |x(s+\theta)|^2\mathrm{d}\mu(\theta)\mathrm{d}s &= \int_0^t\left[\int_{-\infty}^{-s}|x(s+\theta)|^2\mathrm{d}\mu(\theta)+\int_{-s}^0|x(s+\theta)|^2\mathrm{d}\mu(\theta)\right]\mathrm{d}s\\ &= \int_0^t\mathrm{d}s\int_{-\infty}^{-s}\mathrm{e}^{2r(s+\theta)}|x(s+\theta)|^2\mathrm{e}^{-2r(s+\theta)}\mathrm{d}\mu(\theta)+\int_{-t}^0\mathrm{d}\mu(\theta)\int_{-\theta}^t|x(s+\theta)|^2\mathrm{d}s\\ &\le \|\xi\|_{C_r}^2\int_0^t\mathrm{e}^{-2rs}\mathrm{d}s\int_{-\infty}^0\mathrm{e}^{-2r\theta}\mathrm{d}\mu(\theta)+\int_{-\infty}^0\mathrm{d}\mu(\theta)\int_0^t|x(s)|^2\mathrm{d}s\\ &\le \|\xi\|_{C_r}^2\mu_r t+\int_0^t|x(s)|^2\mathrm{d}s.\end{aligned}$$

Consequently,

$$\mathbb{E}V(x(t))\le\mathbb{E}V(\xi(0))+\left(K+n^2\mu_r\mathbb{E}\|\xi\|_{C_r}^2\right)t:=K_t.$$

Let $t=\tau_k\wedge T$. We obtain that

$$\mathbb{E}V(x(\tau_k\wedge T))\le\mathbb{E}V(\xi(0))+\left(K+n^2\mu_r\mathbb{E}\|\xi\|_{C_r}^2\right)T:=K_T.$$

By the definition of τ_k, $x_i(\tau_k)=k$ or $1/k$ for some $i=1,2,\dots,n$,

$$\begin{aligned}\mathbb{P}(\tau_k\le T)[u(k^{-1})\wedge u(k)] &\le \mathbb{P}(\tau_k\le T)V(x(\tau_k\wedge T))\\ &\le \mathbb{E}V(x(\tau_k\wedge T))\\ &\le K_T,\end{aligned}$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}(\tau_k\le T)\le\lim_{k\to\infty}\frac{K_T}{u(k^{-1})\wedge u(k)}=0.$$

Since $T>0$ is arbitrary, we must have $\mathbb{P}(\tau_\infty<\infty)=0$ as required. □

3. Asymptotic bound properties

In Section 2, we show that the solution of Eq. (1.4) is positive and will not explode in any finite time. This nice positive property allows us to further discuss asymptotic bounded properties for the solution of Eq. (1.4).

Theorem 3.1. *Let assumptions (H1), (H2) hold and $p\in(0,1)$. Then there is a positive constant $K=K(p)$, which is independent of the initial data $\{x(t):t\le 0\}\in C_r$, such that the solution $x(t)$ of Eq. (1.4) has the property that*

$$\limsup_{t\to\infty}\mathbb{E}|x(t)|^p\le K. \tag{3.1}$$

Proof. Define a C^2-function $V:\mathbb{R}_{++}^n\to\mathbb{R}_{++}$ by

$$V(x)=\sum_{i=1}^n x_i^p.$$

For any given $\varepsilon\in(0,2r)$, applying the Itô formula to $\mathrm{e}^{\varepsilon t}V(x(t))$ and taking expectation yields

$$\mathrm{e}^{\varepsilon t}\mathbb{E}V(x(t))=\mathbb{E}V(\xi(0))+\mathbb{E}\int_0^t\mathrm{e}^{\varepsilon s}\left[\mathcal{L}V(x(s),x_s)+\varepsilon V(x(s))\right]\mathrm{d}s, \tag{3.2}$$

where $\mathcal{L}V:\mathbb{R}_{++}^n\times C_r\to\mathbb{R}$ is defined by

$$\mathcal{L}V(x,\varphi)=\sum_{i=1}^n px_i^p\left[b_i+\sum_{j=1}^n a_{ij}x_j+\sum_{j=1}^n\int_{-\infty}^0\varphi_j(\theta)\mathrm{d}\mu(\theta)\right]-\frac{p(1-p)}{2}\sum_{i=1}^n x_i^p\left[\sum_{j=1}^n\sigma_{ij}x_j\right]^2.$$

By hypothesis (H1), we may compute that

$$\begin{aligned}\mathcal{L}V(x,\varphi) &\le \sum_{i=1}^n pb_ix_i^p+\sum_{i=1}^n\sum_{j=1}^n pa_{ij}x_i^px_j\\ &\quad+\sum_{i=1}^n\sum_{j=1}^n pb_{ij}\int_{-\infty}^0 x_i^p\varphi_j(\theta)\mathrm{d}\mu(\theta)-\frac{p(1-p)}{2}\sum_{i=1}^n\sigma_{ii}^2x_i^{p+2}\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} pb_i x_i^p + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[p^2 a_{ij}^2 x_i^{2p} + x_j^2\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[p^2 b_{ij}^2 x_i^{2p} + \int_{-\infty}^{0}\varphi_j^2(\theta)\mathrm{d}\mu(\theta)\right] - \frac{p(1-p)}{2}\sum_{i=1}^{n}\sigma_{ii}^2 x_i^{p+2} \\
&\le H(x) + \frac{n}{2}\left[\int_{-\infty}^{0}|\varphi(\theta)|^2\mathrm{d}\mu(\theta) - \mu_r|x|^2\right] - \varepsilon V(x),
\end{aligned}
$$

where

$$
H(x) = \sum_{i=1}^{n}\left[pb_i + \varepsilon\right]x_i^p + \frac{n(1+\mu_r)}{2}|x|^2 + \frac{p^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}^2 + b_{ij}^2\right]x_i^{2p} - \frac{p(1-p)}{2}\sum_{i=1}^{n}\sigma_{ii}^2 x_i^{p+2}.
$$

Note that $p \in (0, 1)$, which implies that $H(x)$ is bounded in $\mathbb{R}_{++}^n$, namely

$$
K_1 := \sup_{x\in\mathbb{R}_{++}^n} H(x) < \infty,
$$

so we have

$$
\mathrm{e}^{\varepsilon t}\mathbb{E}V(x(t)) \le \mathbb{E}V(\xi(0)) + \mathbb{E}\int_0^t \mathrm{e}^{\varepsilon s}\left[K_1 + \frac{n}{2}\left(\int_{-\infty}^{0}|x(s+\theta)|^2\mathrm{d}\mu(\theta) - \mu_r|x(s)|^2\right)\right]\mathrm{d}s.
$$

By hypothesis (H2), we may also compute that

$$
\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{d}s\int_{-\infty}^{0}|x(s+\theta)|^2\mathrm{d}\mu(\theta) \\
&\quad = \int_0^t \mathrm{e}^{\varepsilon s}\mathrm{d}s\left[\int_{-\infty}^{-s}|x(s+\theta)|^2\mathrm{d}\mu(\theta) + \int_{-s}^{0}|x(s+\theta)|^2\mathrm{d}\mu(\theta)\right] \\
&\quad = \int_0^t \mathrm{e}^{\varepsilon s}\mathrm{d}s\left[\int_{-\infty}^{-s}\mathrm{e}^{2r(s+\theta)}|x(s+\theta)|^2\mathrm{e}^{-2r(s+\theta)}\mathrm{d}\mu(\theta)\right] + \int_{-t}^{0}\mathrm{d}\mu(\theta)\int_0^{t+\theta}\mathrm{e}^{\varepsilon(s-\theta)}|x(s)|^2\mathrm{d}s \\
&\quad \le \|\xi\|_{C_r}^2\int_0^t \mathrm{e}^{(\varepsilon-2r)s}\mathrm{d}s\int_{-\infty}^{0}\mathrm{e}^{-2r\theta}\mathrm{d}\mu(\theta) + \int_{-\infty}^{0}\mathrm{e}^{-\varepsilon\theta}\mathrm{d}\mu(\theta)\int_0^t \mathrm{e}^{\varepsilon s}|x(s)|^2\mathrm{d}s \\
&\quad \le \mu_r\|\xi\|_{C_r}^2 t + \mu_r\int_0^t \mathrm{e}^{\varepsilon s}|x(s)|^2\mathrm{d}s.
\end{aligned}
$$

Therefore

$$
\mathrm{e}^{\varepsilon t}\mathbb{E}V(x(t)) \le \mathbb{E}V(\xi(0)) + \varepsilon^{-1}K_1\mathrm{e}^{\varepsilon t} + 0.5n^2\mu_r\|\xi\|_{C_r}^2 t.
$$

This immediately implies that

$$
\limsup_{t\to\infty}\mathbb{E}V(x(t)) \le \varepsilon^{-1}K_1.
$$

Since

$$
|x|^p \le n^{p/2}\max_{1\le i\le n}x_i^p \le n^{p/2}V(x),
$$

we therefore finally have

$$
\limsup_{t\to\infty}\mathbb{E}|x(t)|^p \le K,
$$

and the assertion (3.1) follows by setting $K = n^{p/2}\varepsilon^{-1}K_1$. □

4. Asymptotic pathwise estimation

This section is devoted to derive an asymptotic pathwise estimation of the solution of Eq. (1.4) by the exponential martingale inequality, which shows the solution of Eq. (1.4) how to vary in $\mathbb{R}_{++}^n$.

Theorem 4.1. *Let hypotheses (H1) and (H2) hold. Then, for any positive initial data* $\{x(t) : t \leq 0\} \in C_r$, *the solution* $x(t)$ *of Eq. (1.4) has the property that*

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log(t)} \leq 1 \quad a.s. \tag{4.1}$$

Proof. Define a C^2-function $V : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ by

$$V(x) = \sum_{i=1}^{n} x_i.$$

It is easy to see that

$$\mathrm{d}V(x(t)) = x^{\mathrm{T}}(t)\left(\left[b + Ax(t) + B\int_{-\infty}^{0} x(t+\theta)\mathrm{d}\mu(\theta)\right]\mathrm{d}t + \sigma x(t)\mathrm{d}w(t)\right).$$

Let $\varepsilon \in (0, r)$ be arbitrary. By the Itô formula we can show that

$$\begin{aligned} \mathrm{e}^{\varepsilon t}\log\big(V(x(t))\big) &= \log\big(V(\xi(0))\big) + \int_0^t \mathrm{e}^{\varepsilon s}\big[\varepsilon \log V(x(s))\mathrm{d}s + \mathrm{d}(\log V(x(s)))\big] \\ &= \log\big(V(\xi(0))\big) + \int_0^t \mathrm{e}^{\varepsilon s}\left[\varepsilon \log V(x(s)) + \frac{\mathcal{L}V(x(s), x_s)}{V(x(s))} - \frac{|x^{\mathrm{T}}(s)\sigma x(s)|^2}{2V^2(x(s))}\right]\mathrm{d}s + M(t), \end{aligned} \tag{4.2}$$

where $\mathcal{L}V : \mathbb{R}^n_{++} \times C_r \to \mathbb{R}$ is defined by

$$\mathcal{L}V(x, \varphi) = x^{\mathrm{T}}\left[b + Ax + B\int_{-\infty}^{0} \varphi(\theta)\mathrm{d}\mu(\theta)\right],$$

and

$$M(t) = \int_0^t \mathrm{e}^{\varepsilon s}\frac{|x^{\mathrm{T}}(s)\sigma x(s)|}{V(x(s))}\mathrm{d}w(s)$$

is a real-valued continuous local martingale vanishing at $t = 0$. By the elementary inequality

$$V^2(x)/n \leq |x|^2 \leq nV^2(x) \quad \text{for any } x \in \mathbb{R}^n_{++}, \tag{4.3}$$

we have

$$\begin{aligned} \frac{\mathcal{L}V(x, \varphi)}{V(x)} &\leq \frac{|x|\left[|b| + \|A\||x| + \|B\| \int_{-\infty}^{0} |\varphi(\theta)|\mathrm{d}\mu(\theta)\right]}{V(x)} \\ &\leq \sqrt{n}\left[|b| + \|A\||x| + \|B\| \int_{-\infty}^{0} |\varphi(\theta)|\mathrm{d}\mu(\theta)\right] \end{aligned} \tag{4.4}$$

and

$$\frac{|x^{\mathrm{T}}\sigma x|^2}{V^2(x)} \geq \frac{\hat{\sigma}^2|x|^4}{V^2(x)} \geq \frac{\hat{\sigma}^2|x|^2}{n}, \tag{4.5}$$

where $\hat{\sigma} = \min_{1\leq i\leq n}\sigma_{ii} > 0$. For every integer $k \geq 1$, $0 < p < 1$ and $\delta > 1$, by the exponential martingale inequality, we get that

$$\mathbb{P}\left\{\sup_{0\leq t\leq k}\left[M(t) - \frac{p}{2\mathrm{e}^{\varepsilon k}}\int_0^t \mathrm{e}^{2\varepsilon s}\frac{|x^{\mathrm{T}}(s)\sigma x(s)|^2}{V^2(x(s))}\mathrm{d}s\right] \geq \frac{\delta \mathrm{e}^{\varepsilon k}\log k}{p}\right\} \leq \frac{1}{k^{\delta}}.$$

Since the series $\Sigma_{k=1}^{\infty} k^{-\delta}$ converges, the well-known Borel–Cantelli lemma yields that there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$ there exists an integer $k_0(\omega)$, when $k \geq k_0(\omega)$, and $k - 1 \leq t \leq k$,

$$M(t) \leq \frac{p}{2}\int_0^t \mathrm{e}^{\varepsilon s}\frac{|x^{\mathrm{T}}(s)\sigma x(s)|^2}{V^2(x(s))}\mathrm{d}s + \frac{\delta \mathrm{e}^{\varepsilon(t+1)}\log(t+1)}{p}. \tag{4.6}$$

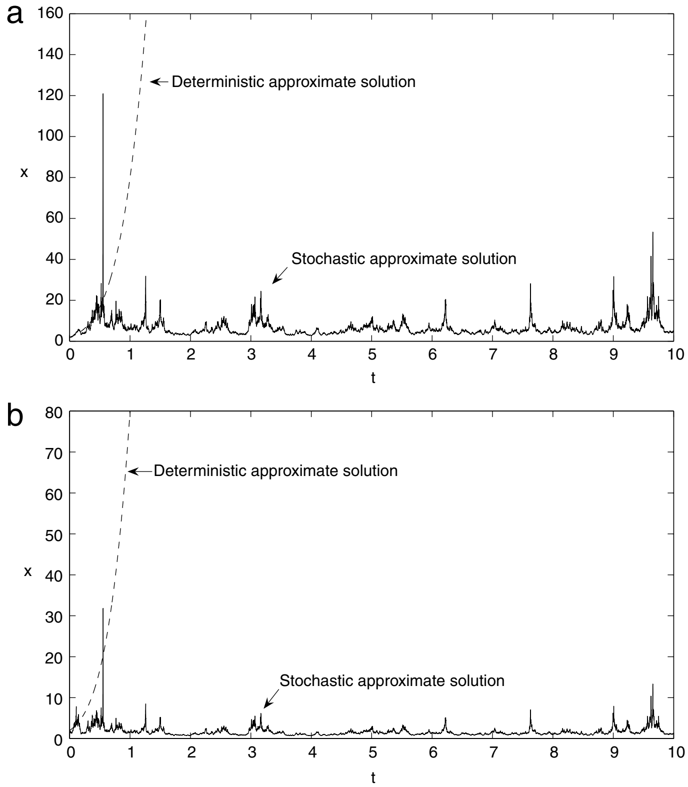

Fig. 1. In graph (a) the solid curve shows a stochastic trajectory generated by the scheme (5.2) for time step $\Delta t = 10^{-4}$, and $\sigma = 0.25$ for a scalar system (5.1) with $A = 0$, $b = B = 1$, $\theta = 0.5$. The corresponding deterministic trajectory is shown by the dashed curve. In graph (b) $\sigma = 1.0$.

Substituting inequalities (4.4), (4.5), (4.6) into (4.2), and letting t sufficiently large, it is obtained almost surely that

$$\begin{aligned} e^{\varepsilon t} \log\big(V(x(t))\big) \le {} & \log\big(V(\xi(0))\big) + p^{-1}\delta e^{\varepsilon(t+1)} \log(t+1) + \int_0^t e^{\varepsilon s} I(x(s))\mathrm{d}s \\ & + \sqrt{n}\|B\| \int_0^t e^{\varepsilon s}\left[\int_{-\infty}^0 |x(s+\theta)|\mathrm{d}\mu(\theta) - \mu_r |x(s)|\right]\mathrm{d}s, \end{aligned} \tag{4.7}$$

where

$$I(x) = \varepsilon \log V(x) + \sqrt{n}b + \sqrt{n}\big(\|A\| + \mu_r\|B\|\big)|x| - 0.5(1-p)n^{-1}\hat{\sigma}^2|x|^2.$$

Note that $I(x)$ is bounded in $\mathbb{R}^n_{++}$, namely

$$K := \sup_{x \in \mathbb{R}^n_{++}} I(x) < \infty.$$

On the other hand, we may compute that

$$\begin{aligned} \int_0^t e^{\varepsilon s}\mathrm{d}s \int_{-\infty}^0 |x(s+\theta)|\mathrm{d}\mu(\theta) &= \int_0^t e^{\varepsilon s}\mathrm{d}s\left[\int_{-\infty}^{-s} |x(s+\theta)|\mathrm{d}\mu(\theta) + \int_{-s}^0 |x(s+\theta)|\mathrm{d}\mu(\theta)\right] \\ &= \int_0^t e^{\varepsilon s}\mathrm{d}s\left[\int_{-\infty}^{-s} e^{r(s+\theta)}|x(s+\theta)|e^{-r(s+\theta)}\mathrm{d}\mu(\theta)\right] + \int_{-t}^0 \mathrm{d}\mu(\theta)\int_0^{t+\theta} e^{\varepsilon(s-\theta)}|x(s)|\mathrm{d}s \\ &\le \|\xi\|_{C_r}\int_0^t e^{(\varepsilon-r)s}\mathrm{d}s \int_{-\infty}^0 e^{-r\theta}\mathrm{d}\mu(\theta) + \int_{-\infty}^0 e^{-\varepsilon\theta}\mathrm{d}\mu(\theta)\int_0^t e^{\varepsilon s}|x(s)|\mathrm{d}s \\ &\le \mu_r\|\xi\|_{C_r} t + \mu_r \int_0^t e^{\varepsilon s}|x(s)|\mathrm{d}s. \end{aligned}$$

It therefore follows from (4.7) that

$$e^{\varepsilon t}\log\big(V(x(t))\big) \le \log\big(V(\xi(0))\big) + p^{-1}\delta e^{\varepsilon(t+1)}\log(t+1) + \varepsilon^{-1}K e^{\varepsilon t} + \mu_r\|\xi\|_{C_r} t$$

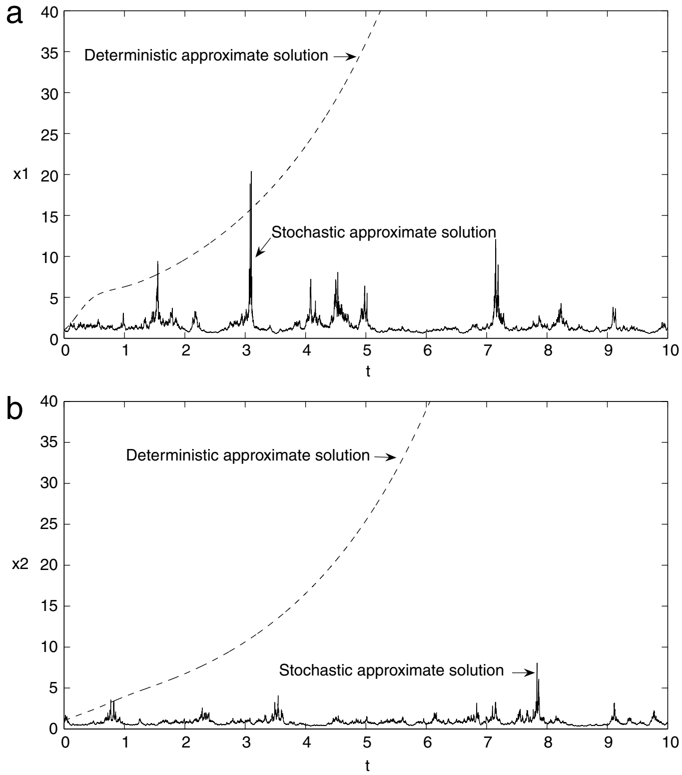

Fig. 2. In both graphs the solid curve represents a stochastic approximate trajectory for system (5.3) generated by the scheme (5.2) with time step $\Delta t = 10^{-4}$, $\varepsilon = 1.0$ and $\theta = 0.5$, whilst the corresponding deterministic approximate solution is shown by the dashed curve. Graph (a) shows the first component x_1 and graph (b) the second, x_2.

for $k-1 \le t \le k$ and $k \ge k_0(\omega)$ whenever $\omega \in \Omega_0$. This implies

$$\limsup_{t\to\infty} \frac{\log(V(x(t)))}{\log(t)} \le \frac{\delta e^{\varepsilon}}{p} \quad a.s.$$

Noting inequality (4.3), letting $\delta \to 1$, $p \to 1$ and $\varepsilon \to 0$, we obtain that

$$\limsup_{t\to\infty} \frac{\log(|x(t)|)}{\log(t)} \le 1 \quad a.s.$$

as required. □

5. Examples and computer simulations

In this section, we explore system behaviour using numerical solutions of the stochastic Lotka–Volterra system (1.4). For convenience, let the probability measure $\mu(\theta)$ be e^{θ} on $(-\infty, 0]$. So the stochastic Lotka–Volterra system (1.4) will be written as

$$dx(t) = \mathrm{diag}\big(x(t)\big)\left[\left(b + Ax(t) + e^{-t}B\int_{-\infty}^{0} e^{s}\xi(s)ds + e^{-t}B\int_{0}^{t} e^{s}x(s)ds\right)dt + \sigma x(t)dw(t)\right], \tag{5.1}$$

where $t \ge 0$ and $\mathrm{diag}\big(x(t)\big) = \mathrm{diag}\big(x_1(t), \ldots, x_n(t)\big)$, $x(t) = \xi(t)$ as $t \le 0$ is known exactly. We employ the Euler scheme to discretize such equation, where the integral term is approximated by using the composite θ-rule as a quadrature (cf. [16]). In particular, for $t = \Delta t, 2\Delta, \ldots$, we may obtain the discrete approximate solution with respect to (5.1)

$$\begin{aligned} y((k+1)\Delta) = {} & y(k\Delta) + \mathrm{diag}\big(y(k\Delta)\big)\Bigg[\Delta\Bigg(b + Ay(k\Delta) + e^{-k\Delta}B\int_{-\infty}^{0} e^{s}\xi(s)ds \\ & + \Delta e^{-k\Delta}B\sum_{j=0}^{k}\omega_j^{(k)}e^{j\Delta}y(j\Delta)\Bigg) + \sigma y(k\Delta)\Delta w_k\Bigg], \end{aligned} \tag{5.2}$$

where $\Delta w_k = w((k+1)\Delta) - w(k\Delta)$, $k = 0, 1, 2, \ldots$, the general composite θ-rule has weights

$$\{\omega_0^{(k)}, \omega_1^{(k)}, \ldots, \omega_{k-1}^{(k)}, \omega_k^{(k)}\} = \{\theta, 1, \ldots, 1, 1-\theta\}, \quad 0 \le \theta \le 1$$

and $\sum_{j=0}^{k} \omega_j^{(k)} = n, n \ge 0$.

From the above scheme (5.2) for a scalar example of system (5.1) with $A = 0, b = B = 1, \theta = 0.5$, the initial data $\xi(s) = s^2 + 2$ for $s \le 0$ and $\Delta t = 10^{-4}$, we may obtain the simulation Fig. 1 which shows that Eq. (5.1) has a unique positive solution. In each case the corresponding prediction of the deterministic model, which explodes at finite time, is also shown. Their simulations illustrate the result of this paper, namely that environmental noise suppresses population explosion in such systems. Moreover, comparison of Fig. 1(a) and (b) suggests that fluctuations reduce as the noise level increases.

Finally, consider the bivariate system

$$\begin{aligned} \mathrm{d}x_1(t) &= x_1(t)\left(1 - x_1(t) + 2\int_{-\infty}^{0} x_2(t+s)\mathrm{d}e^s\right)\mathrm{d}t + \varepsilon x_1^2(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) &= x_2(t)\left(1 - 2x_2(t) + 2\int_{-\infty}^{0} x_1(t+s)\mathrm{d}e^s\right)\mathrm{d}t + 2\varepsilon x_2^2(t)\mathrm{d}w_2(t), \end{aligned} \tag{5.3}$$

with the initial data $\xi_1(s) = e^{-0.5s}$, $\xi_2(s) = s^2 + 1$ for $s \le 0$. Fig. 2 shows a realization of the numerical solution of this system based on the above scheme (5.2), with step $\Delta t = 10^{-4}$ and noise level $\varepsilon = 1.0$. Comparison with the deterministic solution supports our result, namely that noise suppresses the population explosion.

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